

Spatial Geometry of Non-Abelian Gauge Theory in $2 + 1$ Dimensions

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Abstract

The Hamiltonian dynamics of $2 + 1$ dimensional Yang-Mills theory with gauge group $SU(2)$ is reformulated in gauge invariant, geometric variables, as in earlier work on the $3 + 1$ dimensional case. Physical states in electric field representation have the product form $\Psi_{\text{phys}}[E^{ai}] = \exp(i\Omega[E]/g)F[G_{ij}]$, where the phase factor is a simple local functional required to satisfy the Gauss law constraint, and G_{ij} is a dynamical metric tensor which is bilinear in E^{ak} . The Hamiltonian acting on $F[G_{ij}]$ is local, but the energy density is infinite for degenerate configurations where $\det G(x)$ vanishes at points in space, so wave functionals must be specially constrained to avoid infinite total energy. Study of this situation leads to the further factorization $F[G_{ij}] = F_c[G_{ij}]\mathcal{R}[G_{ij}]$, and the product $\Psi_c[E] \equiv \exp(i\Omega[E]/g)F_c[G_{ij}]$ is shown to be the wave functional of a topological field theory. Further information from topological field theory may illuminate the question of the behavior of physical gauge theory wave functionals for degenerate fields.

1 Introduction

The purpose of this paper is to apply a recently developed [1] spatial-geometric approach to $SU(2)$ and $SU(3)$ Yang-Mills theory in $3 + 1$ dimensions to the case of the $SU(2)$ gauge theory in $2 + 1$ dimensions. The Hamiltonian for states satisfying the Gauss law constraint is simpler in the $2 + 1$ dimensional case, and we can begin to consider its physical implications.

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The approach combines the following two ideas: 1) Hamiltonian dynamics can be reformulated in gauge invariant variables, with the Gauss law constraint automatically satisfied, and 2) the basic equations of the canonical formalism, except the definition of H , are invariant under diffeomorphisms of the spatial domain.

Using these ideas one can show that physical states in electric field representation [2] take the form

$$\Psi [E^{ai}] = e^{i\Omega[E]/g} F [G_{ij}] \quad (1.1)$$

where $\Omega[E]$ is an explicit local functional of $E^{ai}(x)$ required by the constraint, and $G_{ij}(x)$ is the positive gauge invariant variable

$$G_{ij} = \epsilon_{ik}\epsilon_{jl}E^{ak}E^{al} \quad (1.2)$$

which transforms as a covariant 2-tensor under diffeomorphisms.

We derive an expression for the expectation value (or matrix element) of the Hamiltonian in which the phase factor of (1.1) cancels. In such matrix elements

$$\langle F [G_{ij}] | H | F [G_{ij}] \rangle \quad (1.3)$$

- a) All gauge indices completely contract out.
- b) H contains covariant spatial derivatives ∇_i with the Christoffel connection $\Gamma_{ij}^k(G)$ of the dynamical metric G_{ij} , and the curvature scalar $R(G)$ also appears. This means that the underlying spatial geometry is purely Riemannian. (This was also true [1] for gauge group $SU(2)$ in $3+1$ dimensions, while for $SU(3)$ in $3+1$ dimensions a more complicated geometry with torsion was found).
- c) H also contains the fixed Cartesian metric δ_{ij} of \mathbb{R}^2 . So H is *not* diffeomorphism invariant, but there is a clear separation of invariant and non-invariant parts.
- d) The resolution of the Gauss law constraint requires the $1/g$ factor in (1.1) when the usual “perturbative normalization” of A and E is used, and H also contains $1/g$ and $1/g^2$ terms, as well as positive powers up to g^2 . It is therefore hard to see how to apply perturbation theory and the strong coupling expansion is also problematic.

What has been achieved, therefore, is a reduction of the gauge theory to the subspace of physical states, where we find a local non-linear Hamiltonian

in the three components of $G_{ij}(x)$ rather than the initial six components of $E^{ai}(x)$. But this result may well be of only formal significance if appropriate dynamical methods cannot be found.

There is one aspect of the new formulation which indicates that the geometric structure may have physical implications. Specifically, the Hamiltonian is singular for degenerate configurations in which $\det G(x)$ vanishes in the spatial domain. At the points of degeneracy the rank of 3×2 electric field matrix $E^{ai}(x)$ is less than two. Generically degeneracy occurs at isolated points in \mathbb{R}^2 . Since H is the transform of the standard form $\int d^2x (E^2 + B^2)$, the singularities are repulsive, and a variational trial functional or candidate Schrödinger eigenfunction which is not specially behaved for degenerate fields will have infinite energy. This does not necessarily mean that wave functionals vanish, because some singularities are due to choice of variables and are resolved without physical consequence. However in this case the singularity originates in the phase $\Omega[E]$ which is required by the Gauss law. Heuristic arguments indicate that the singularities are significant, but do not prove that wave functions vanish. However, we are led to examine the situation more carefully in the context of a physical picture based on an analogy with the centrifugal barrier in quantum mechanics.

It is a familiar fact that eigenfunctions for angular momentum $\ell \neq 0$ in a central potential take the form

$$\Psi_\ell^m(\vec{x}) = Y_\ell^m(\hat{x}) r^\ell R(r) \quad (1.4)$$

of a product of spherical harmonic, the centrifugal factor r^ℓ and a regular function. Our considerations suggest the analogous product form

$$\Psi[E] = e^{i\Omega[E]/g} F_c[G] \mathcal{R}[G] \quad (1.5)$$

for all physical wave functionals in the non-abelian gauge theory in which the centrifugal functional $F_c[G]$ carries effects of the singularities of H , and the residual factor $\mathcal{R}[G]$ is presumably unconstrained. It should be pointed out that there is no analogue in the field theory of $\ell = 0$ states which are not suppressed at $r = 0$. All physical states carry the phase factor in (1.5).

Comparison of the Hamiltonian (1.3) with its quantum mechanical analogue motivates the definition of $F_c[G]$ as the solution of a set of functional differential equations which are exactly the diffeomorphism and Wheeler de Witt constraint equations of (the Euclidean continuation of) $2 + 1$ dimensional gravity. It then turns out that the product of the first two factors in (1.5) satisfies the constraint equations of the Chern-Simons (or topological

b/F), form [3] of the gravity theory. The solution of these constraints can be written as the path integral representation

$$\begin{aligned}\Psi_c[E] &\equiv e^{i\Omega[E]/g} F_c[G] \\ &= \int [dU(x)] \exp \left\{ \frac{1}{g} \int d^2x \operatorname{Tr} (E^i U^{-1} \partial_i U) \right\}\end{aligned}\quad (1.6)$$

in which $U(x)$ is an $SU(2)$ matrix and $e^i(x) = T^a E^{ai}(x)$. The phase factor can be extracted from (1.6) and another path integral representation written for the real gauge invariant factor $F_c[G]$; see (5.21) below.

The key question is whether $\Psi_c[E]$ vanishes for degenerate fields. Despite considerable study of the representation (1.6) we have not so far been able to answer this question. Another approach is to insert the product representation (1.5) in the Hamiltonian, and examine the constraints on the residual factor $\mathcal{R}[G]$ for singular metrics. The result is that the Hamiltonian governing $\mathcal{R}[G]$ is considerably less singular than the form (1.3), so that $\mathcal{R}[G]$ need not vanish for degenerate fields whether or not $F_c[G]$ vanishes.

At present the conclusions are rather ambiguous. Heuristic analysis of the singularities of H suggests that physical wave functions $\Psi[E]$ vanish for degenerate fields, but this is not confirmed by further investigation. Instead there is an apparently consistent scenario in which the factor $\Psi_c[E]$ of (1.5) resolves the singularities of H without the requirement that either $\Psi_c[E]$ or $\mathcal{R}[G]$ vanish. Thus the factor $\Psi_c[E]$ plays an important role in both scenarios, and it is very curious that the wave functional of a topological theory enters into the analysis of a non-trivial dynamical theory. Of course, the explicit form of $\Psi_c[E]$ should settle the issue of its vanishing, and it is to be hoped that there is now sufficient knowledge of two dimensional and topological field theory to make progress on this.

The previous discussion raises the question whether any similar situation is expected in $3+1$ dimensions. The Hamiltonian of [1] is considerably more complicated, but it is again singular for degenerate configurations of a tensor variable bilinear in the electric field. A careful study of the significance of these singularities is required. However even a cursory inspection of the Hamiltonian shows that physical wave functionals naturally have the product structure (1.5) with a prefactor $\Psi_c[E]$ which satisfies the constraint equations of a 4-dimensional topological b-F theory [19], and that the solution of these equations is just (1.6) again, but with integration over \mathbb{R}^3 instead of \mathbb{R}^2 .

The first gauge invariant formulation of Yang-Mills theory was obtained by Halpern for the self dual theory, and a metric tensor also appeared in

his work [4]. The gauge non-invariant metric tensor $G_{\mu\nu} = A_\mu^a A_\nu^a$ was used as the effective field variable in the long distance limit by Ne’eman and Sijacki [5]. Lunev has developed a geometric formulation of the Lagrangian form of non-abelian gauge theories both for $2 + 1$ and $3 + 1$ dimensions [6]. Discussions of a gauge field geometry with torsion have also appeared recently [7], [8], and there is a geometric formulation of the Hamiltonian dynamics of $3 + 1$ dimensional $SU(2)$ gauge theory which uses the potential representation [9].

Variational calculations for gauge theory in which the Gauss law constraint is enforced by averaging the gauge group have been presented by Kogan and Kovner [10]. We also list here some other references to recent studies of the non-perturbative physics of non-abelian gauge theories in $2 + 1$ and $3 + 1$ dimensions [11–16].

2 Spatial Geometry of the Gauge Theory

2.1 The Canonical Formalism in E -field Representation

The action¹ of $SU(2)$ gauge theory in $2 + 1$ dimensional flat Minkowski space is

$$S = -\frac{1}{4g^2} \int d^2x dt (F_{\mu\nu}^a)^2. \quad (2.1)$$

The coupling constant has dimensions of inverse length, $[g] = 1$. This gives a theory which is super-renormalizable in perturbation theory, but there are still unresolved non-perturbative issues [17], namely the questions of confinement and generation of a mass gap.

We wish to set up the canonical formalism in $A_0^a = 0$ gauge. The canonical momentum is $E^{ai}(x) = \delta S / \delta \dot{A}_i^a(x) = -F^{a(0i)} / g^2$, and the canonical commutation rule, Gauss law constraint, magnetic field and Hamiltonian are

$$\left[A_j^a(x), E^{bk}(x') \right] = i\delta^{ab}\delta_j^k\delta^{(2)}(x - x') \quad (2.2)$$

$$G^a(x)\psi = \left(\partial_i E^{ai}(x) + \epsilon^{abc} A_i^b(x) E^{ci}(x) \right) \psi = 0 \quad (2.3)$$

$$B^a(x) = \epsilon^{ij} \left(\partial_i A_j^a + \frac{1}{2} \epsilon^{abc} A_i^b A_j^c \right) \quad (2.4)$$

¹The perturbative “normalization” of gauge fields was implicitly used in Sec. 1, but we now use the normalization in which $1/g^2$ appears as a factor in the action. The scaling $A \rightarrow gA$ and $E \rightarrow E/g$ may be used to change any formula in Sections 2.1–6 to perturbative normalization.

$$H = \frac{1}{2} \int d^2x \left\{ g^2 \delta_{ij} E^{ai} E^{aj} + \frac{1}{g^2} B^a B^a \right\} \quad (2.5)$$

Only the definition of H requires the Cartesian spatial metric δ_{ij} , and (2.2–2.4) are covariant under diffeomorphisms of the spatial domain \mathbb{R}^2 , that is coordinate transformations $x^i \rightarrow y^\alpha(x^i)$, $i, \alpha = 1, 2$ and transformation rules

$$\begin{aligned} A^a{}_i(x) &\rightarrow A^a{}_\alpha(y) = \frac{\partial x^i}{\partial y^\alpha} A^a{}_i(x) \\ E^{ai}(x) &\rightarrow E^{a\alpha}(y) = \left| \frac{\partial x}{\partial y} \right| \frac{\partial y^\alpha}{\partial x^i} E^{ai}(x) \end{aligned} \quad (2.6)$$

from which we see that A is a covariant vector (a 1-form) and E is a contravariant vector density.

The dynamical problem of gauge theory is to find solutions of the functional Schrödinger equation $H\psi = E\psi$ for states which satisfy (2.3). We note that H is not diffeomorphism invariant, because the fixed metric δ_{ij} appears and because both terms have density weight two in the dynamical variables, whereas weight one is required for invariance. Nevertheless we shall be guided in our work by the idea of preserving the diffeomorphism covariance of the canonical formalism.

In almost all work on the Hamiltonian formalism in gauge theory, the potential representation is used in which $\psi \rightarrow \psi[A]$ and the electric field acts by functional differentiation $E^{aj} = -i\delta/\delta A^a{}_j$. However the implementation of Gauss' law (2.3) leads either to a non-local Hamiltonian [9, 13] or to averaging over the gauge group using additional variables [10]. We therefore use the electric field representation [2] obtained by the canonical Fourier transformation

$$\psi[A] = \int [dE^{ai}(x)] \exp \left\{ i \int d^2x A^a{}_i(x) E^{ai}(x) \right\} \Psi[E]. \quad (2.7)$$

In this representation it is $A^a{}_j = i\delta/\delta E^{aj}$ which acts by differentiation.

We now consider the gauge transformation by the 3×3 orthogonal matrix $T^{ab}(x)$, which acts as

$$\begin{aligned} A^a{}_i &\rightarrow T A^a{}_i = \frac{1}{2} \epsilon^{abc} T^{bd} \partial_i T^{cd} + T^{ab} A^b{}_i \\ E^{ai} &\rightarrow T E^{ai} = T^{ab} E^{bi} \end{aligned} \quad (2.8)$$

The Gauss law (2.3) requires $\psi[TA] = \psi[A]$ and this gives [2]

$$\psi[TE] = \exp -i \int d^2x \frac{1}{2} \epsilon^{abc} E^{ai} \left(T^{db} \partial_i T^{dc} \right) \psi[E]. \quad (2.9)$$

It is the fact that the convective term in the Gauss law is a multiplication operator in E -field representation, viz

$$G^a(x)\psi[E] = \frac{1}{g} \left(\partial_i E^{ai}(x) - i\epsilon^{abc} E^{bc}(x) \frac{\delta}{\delta E^{ci}(x)} \right) \psi[E] \quad (2.10)$$

that leads to the phase factor in (2.9).

2.2 The Unitary Transformation

In the same spirit as in [2], but with some differences, we implement (2.9) by a unitary transformation

$$\psi[E] = \exp i\Omega[E] F[E] \quad (2.11)$$

in which the phase factor $\exp i\Omega[E]$ is the intertwining operator which removes the convective term from the Gauss law generator:

$$\begin{aligned} G^a(x) \exp i\Omega[E] &= \exp i\Omega[E] \overline{G}^a(x) \\ \overline{G}^a(x) &\equiv -i\epsilon^{abc} E^{bj}(x) \frac{\delta}{\delta E^{cj}(x)} \end{aligned} \quad (2.12)$$

It is clear that (2.9) is satisfied by any functional $\Omega[E]$ with the gauge transformation property

$$\Omega[TE] = \Omega[E] - \frac{1}{2} \int d^2x E^{ai} \epsilon^{abc} T^{db} \partial_i T^{dc} \quad (2.13)$$

and we can see that (2.12) is also satisfied by looking at the form of (2.13) for infinitesimal gauge transformations. We also require that $\Omega[E]$ be invariant under spatial diffeomorphisms.

The phase $\Omega[E]$ which is introduced to satisfy Gauss' law is also the key to the spatial geometric properties of our approach to gauge theory. Most of these properties can be deduced directly from the gauge (2.13) and diffeomorphism requirements for $\Omega[E]$, rather than from any specific form. Nevertheless it is useful to exhibit the following simple local functional which is easily shown to satisfy both requirements:

$$\Omega[E] = \int d^2x \epsilon^{abc} \left(E^{ai} E^{bj} \partial_i E^{ck} \right) \varphi_{jk} \quad (2.14)$$

where $\varphi^{ij} = E^{ai} E^{aj}$ is a gauge invariant tensor density and φ_{jk} is its matrix inverse (i.e., $\varphi^{ij} \varphi_{jk} = \delta^i_k$). One should note that the two requirements

do not specify $\Omega[E]$ uniquely, but that any two solutions (e.g., $\Omega[E]$ and $\Omega'[E]$) must differ by a gauge and diffeomorphism invariant functional. For example, one could take

$$\Omega'[E] = \Omega[E] + c \int d^2x (\det \varphi^{mn})^{1/2} \quad (2.15)$$

and there are many other possibilities. It turns out that the choice (2.14) gives the simplest spatial geometry in a way which will make precise below, but we now resume the general discussion which is independent of any specific choice.

The next steps in the development of the geometry are

1) implementation of the unitary transformation on the operators of the theory, specifically

$$\begin{aligned} \overline{E}^{ai}(x) &= e^{-i\Omega[E]} E^{ai}(x) e^{i\Omega[E]} \\ \overline{A}^a{}_i(x) &= e^{-i\Omega[E]} A^a{}_i(x) e^{i\Omega[E]}, \end{aligned} \quad (2.16)$$

2) study of the residual Gauss constraint on $F[E]$,

$$\overline{G}^a(x) F[E] = 0 \quad (2.17)$$

which implies $F[TE] = F[E]$ for finite gauge transformations, and

3) expression of the Hamiltonian in geometric variables.

The transformed operators are

$$\begin{aligned} \overline{E}^{aj}(x) &= E^{aj}(x) \\ \overline{A}^a{}_j(x) &= i \frac{\delta}{\delta E^{aj}(x)} - \frac{\delta \Omega}{\delta E^{aj}(x)} \\ &\equiv i \frac{\delta}{\delta E^{aj}(x)} + \omega^a{}_j(x). \end{aligned} \quad (2.18)$$

The quantity $\omega^a{}_i(x)$ is a covariant vector if $\Omega[E]$ is diffeomorphism invariant, while (2.13) tells us that

$$\omega^a{}_i \rightarrow \frac{1}{2} \epsilon^{abc} T^{bd} \partial_i T^{cd} + T^{ab} \omega^b{}_i \quad (2.19)$$

under the gauge transformation $E^{ai} \rightarrow T^{ab} E^{bi}$. These are exactly the properties of an $SU(2)$ gauge connection, so the unitary transformation produces the composite gauge connection $\omega^a{}_i(x)$ which depends on the electric field.

If the specific phase $\Omega[E]$ of (2.14) is used, we see that $\omega^a_i(x)$ is a local function of E and its first derivatives.

The transformed magnetic field is

$$\begin{aligned}\overline{B}^a &= \epsilon^{jk} \left(\partial_j \overline{A}_k + \frac{1}{2} \epsilon^{abc} \overline{A}_j^b \overline{A}_k^c \right) \\ &= \widehat{B}^a + i \epsilon^{jk} \widehat{D}_j \frac{\delta}{\delta E^{ak}} - \frac{1}{2} \epsilon^{jk} \epsilon^{abc} \frac{\delta^2}{\delta E^{bj} \delta E^{ck}}\end{aligned}\quad (2.20)$$

where (2.18) has been used to get the second line which contains the gauge covariant derivative with connection ω and the composite magnetic field

$$\widehat{D}_i \frac{\delta}{\delta E^{aj}} \equiv \partial_i \frac{\delta}{\delta E^{aj}} + \epsilon^{abc} \omega^b_i \frac{\delta}{\delta E^c_j} \quad (2.21)$$

$$\widehat{B}^a \equiv \epsilon^{ij} \left(\partial_i \omega^a_j + \frac{1}{2} \epsilon^{abc} \omega^b_i \omega^c_j \right). \quad (2.22)$$

We have dropped the contact term

$$\epsilon^{ij} \epsilon^{abc} \frac{\delta}{\delta E^{bi}(x)} \omega^c_j(x) \quad (2.23)$$

in (2.20), which contains the ill-defined quantities $\delta(0)$ and $\partial_i \delta(0)$ if (2.14) is used. The covariant point splitting argument of [1] shows that this contact term actually vanishes. In general, however, the important issue of the regularization within our approach to gauge theory has not yet been studied.

With \overline{B}^{ai} as given in (2.20) the unitary transform of the Hamiltonian (2.5) may be written as

$$\overline{H} = \frac{1}{2} \int d^2x \left\{ g^2 \delta_{ij} E^{ai} E^{aj} + \frac{1}{g^2} \overline{B}^a \overline{B}^a \right\}. \quad (2.24)$$

This may be regarded as an intermediate result within our approach. It is gauge covariant, because of the placement of the composite connection ω^a_i , but spatial geometric variables have not yet appeared.

Finally we note two useful identities which follow from the definition (2.18) of the composite connection. For an infinitesimal gauge transformation $T^{ab}(x) = \delta^{ab} - \epsilon^{abc} \theta^c(x)$, the gauge requirement (2.13) on the phase $\Omega[E]$ reduces to

$$\int d^2x E^{ai} \left(\partial_i \theta^a + \epsilon^{abc} \omega^b_i \theta^c \right) = 0. \quad (2.25)$$

Partial integration gives the gauge identity

$$\widehat{D}_i E^{ai} \equiv \partial_i E^{ai} + \epsilon^{abc} \omega^b_i E^{ci} = 0. \quad (2.26)$$

We also consider the effect of an infinitesimal diffeomorphism of E parameterized by the vector field $v^i(x)$. The transformation law becomes

$$\delta E^{ai} = (\partial_j v^i) E^{aj} - \partial_j (v^j E^{ai}). \quad (2.27)$$

But $\Omega[E]$ is required to be invariant, so that

$$0 = \int d^2x \frac{\delta \Omega}{\delta E^{ai}} \delta E^{ai} = - \int d^2x v^j [E^{ai} (\partial_i \omega_j - \partial_j \omega_i) + \omega^a_j \partial_i E^{ac}]. \quad (2.28)$$

Inserting the gauge identity (2.25), and using (2.22) one finds the diffeomorphism identity

$$\epsilon_{ij} E^{aj} \hat{B}^a = 0 \quad (2.29)$$

which ensures orthogonality the electric field and the composite magnetic field.

2.3 The functional $F[E]$

Our goal in this section is to solve explicitly the Gauss law constraint for the functional $F[E]$, which according to (2.8,2.17) is simply $F[TE] = F[E]$. If $E = \{E^{ai}(x)\}$ is a configuration of the electric field, the gauge transformed configuration is $TE = \{T^{ab}(x)E^{bi}(x)\}$. This transformation is purely local, and we shall analyze it point by point, without mentioning x explicitly. Then the tensor index of the electric field is just a label. We view E^{a1} and E^{a2} as two vectors in (gauge) 3-space. Classical invariant theory then asserts that the invariants are freely generated by the scalar products $\varphi^{ij} \equiv E^{ai} E^{aj}$. There is a quick proof if one assumes that E^{a1} and E^{a2} are linearly independent, or equivalently that $\det \varphi^{ij} \neq 0$. We apply Schmidt orthogonalization to write $E^{a1} = \lambda_1 n^{a1}$ with $\lambda_1 > 0$ and n^{a1} a unit vector, and then $E^{a2} = \mu_2 n^{a1} + \lambda_2 n^{a2}$ with $\lambda_2 > 0$ and n^{a2} a unit vector orthogonal to n^{a1} . Then $(n^{a1}, n^{a2}, n^{a1} \times n^{a2})$ is an orthonormal frame. The numbers λ_1 , λ_2 and μ_2 are uniquely defined in terms of φ^{ij} and the correspondence is one to one. There is a unique rotation sending n^{a1} to $(1, 0, 0)$ and n^{a2} to $(0, 1, 0)$. This rotation sends E^{a1} to $(\lambda_1, 0, 0)$ and E^{a2} to $(\mu_2, \lambda_2, 0)$. So we have found a canonical representative depending only on the invariants φ^{ij} on each non-degenerate orbit. This is the result we were after. Wilson lines for the composite connection ω^a_i , which are gauge invariant objects, can be expressed solely in terms of φ^{ij} . The point is that the Wilson line is unchanged when one replaces a given electric field configuration by the canonical representative (depending only on φ^{ij}) of its gauge orbit.

To summarize we have shown that the most general gauge invariant functional of the electric field can be rewritten as a functional of φ^{ij} or equivalently of G_{ij} defined by (1.2) which as we shall see later is a more convenient variable. Hence on the physical subspace we can write $F[G_{ij}]$ for the wave functional.

In two dimensions and planar or spherical geometry, the gauge group is connected and to check for gauge invariance it is enough to check infinitesimal gauge invariance (i.e., the Gauss law).

One can use the fact that the general E^{ai} field is the gauge transform of its canonical representative to relate the measures.

$$\prod_{a,i} dE^{ai} \propto d\rho_{\text{Haar}}^{\text{SO}(3)} \prod_{i \leq j} dG_{ij}. \quad (2.30)$$

2.4 Tensor variables

We now wish to introduce a basis e^a_1, e^a_2, e^a in the adjoint representation of $\text{SU}(2)$. These are defined in terms of the electric field by

$$E^{ai} = \epsilon^{ij} e^a_j, \quad \epsilon^{abc} e^a_i e^b_j = \sqrt{G} \epsilon_{ij} e^c. \quad (2.31)$$

It is easy to check that e^a_i is a gauge vector and a covariant vector, that the metric G_{ij} of (1.2) is simply $e^a_i e^a_j$, a gauge invariant symmetric positive covariant tensor whose determinant we denote by G , and that e^a is a (unit) gauge vector and a pseudoscalar. From now on, we use the metric G_{ij} to raise and lower indices, the only exceptions being the epsilon symbols. The three gauge vectors e^a_1, e^a_2 and e^a are linearly independent and satisfy

$$e^a_i e^a = 0, \quad e^{ai} e^a_j = \delta^i_j, \quad \delta^{ab} = e^a e^b + e^{ai} e^b_i. \quad (2.32)$$

Any quantity with gauge indices can be expanded in the semi-orthogonal frame e^a_1, e^a_2, e^a . For gauge covariant quantities, the coefficients will be geometric objects in the spatial geometry of the gauge theory.

2.5 Riemannian geometry

We expand the gauge covariant derivative of the basis vectors. We can write

$$\hat{D}_i e^a_j \equiv \partial_i e^a_j + \epsilon^{abc} \omega^b_i e^c_j = M^l_{ij} e^a_l + N_{ij} e^a \quad (2.33)$$

By computing the effect of a change of coordinates, one sees explicitly that N_{ij} is a tensor and M^l_{ij} an affine connection. The connection is metric

compatible because

$$\partial_i G_{jk} = \hat{D}_i G_{jk} = \hat{D}_i(e_j^a e_k^a) = M_{ij}^l e_l^a e_k^a + M_{ik}^l e_l^a e_j^a = M_{ij}^l G_{kl} + M_{ik}^l G_{jl}. \quad (2.34)$$

From the gauge identity $\hat{D}_i E^{ai} = \epsilon^{ij} \hat{D}_i e_j^a = 0$ we conclude that $\hat{D}_i e_j^a$ is symmetric in ij , and the same is true of M_{ij}^k and N_{ij} . But the only metric compatible symmetric affine connection is the Levi-Civita connection. Hence

$$M_{ij}^k = \Gamma_{ij}^k \equiv \frac{1}{2} G^{kl} (\partial_i G_{lj} + \partial_j G_{il} - \partial_l G_{ij}). \quad (2.35)$$

From $\epsilon^{ij} \hat{D}_i e_j^a = 0$ we also get, by contraction with e_k^a , another form of the gauge identity:

$$e_k^a \epsilon^{ij} \partial_i e_j^a - \sqrt{G} e^a \omega_k^a = 0. \quad (2.36)$$

Using the Leibnitz property of the gauge covariant derivative we compute $\hat{D}_i e^a = -N_i^k e_k^a$. To summarize, we have proved that

$$\begin{aligned} \hat{D}_i e_j^a &= \Gamma_{ij}^k e_k^a + N_{ij} e^a \\ \hat{D}_i e^a &= -N_i^k e_k^a. \end{aligned} \quad (2.37)$$

where Γ_{ij}^k is the Levi-Civita connection and N_{ij} a symmetric tensor.

Up to now, we have only used general properties of the phase $\Omega[E]$ and the outcome is already quite remarkable. But to go further we need an explicit formula. We have chosen the simplest possibility, for which N_{ij} vanishes.

2.6 The explicit phase $\Omega[E]$ and its variation

The formula

$$\Omega[E] = \int d^2x \sqrt{G} e^{ai} \partial_i e^a \quad (2.38)$$

defines a scalar. It is not difficult to check that this definition coincides with (2.14) and has the proper behavior under gauge transformations (2.13).

It is clear that $\Omega[E]$ is a homogeneous function of the electric field of degree 1, but it also has a deeper symmetry, which we call local tensorial homogeneity. Let Λ_j^i be an invertible tensor field of type $(1, 1)$. We can use it to define a local $GL(2)$ transformation on the electric field by $(\Lambda E)^{ai} = \Lambda_j^i E^{aj}$. Using the original formula (2.14) and inserting the frame variables after the transformation, it is easy to check that

$$\Omega[\Lambda E] = - \int d^2x \sqrt{G} \Lambda_j^i e^a \partial_i e^{aj}. \quad (2.39)$$

The infinitesimal version gives, after some reshuffling of indices,

$$\sqrt{G}\omega_i^a e^{ak} = e^a \epsilon^{jk} \partial_i e_j^a, \quad (2.40)$$

an identity which bears a striking resemblance to the gauge identity (2.36). Those two identities fix the scalar product in gauge space of the composite connection with the three basis vectors. One can see explicitly that

$$\sqrt{G}\omega_i^a = \epsilon^{jk} (e_k^a e^b \partial_i e_j^b + e^a e_i^b \partial_j e_k^b) \quad (2.41)$$

and that

$$\begin{aligned} \hat{D}_i e_j^a &= \Gamma_{ij}^k e_k^a \\ \hat{D}_i e^a &= 0 \\ \hat{B}^a &= -\frac{1}{2}\sqrt{G}R e^a. \end{aligned} \quad (2.42)$$

The first two equations show the vanishing of N_{ij} for our choice of phase. The third one comes from the computation of the commutator of the covariant derivative. One finds

$$\left[\hat{D}_i, \hat{D}_j \right] e_k^a = R_{kij}^l e_l^a \quad (2.43)$$

But $\left[\hat{D}_i, \hat{D}_j \right] \lambda^a = \epsilon^{abc} F_{ij}^b \lambda^c = \epsilon^{abc} \epsilon_{ij} \hat{B}^b \lambda^c$. According to the diffeomorphism identity, the composite magnetic field is perpendicular to the electric field. Hence $\hat{B}^b = \hat{B} e^b$ and using various two dimensional tensor identities, such as

$$2R_{kij}^m = -\det G \epsilon_{ij} \epsilon_{kl} G^{lm} R, \quad (2.44)$$

where R is the scalar curvature, and we finally obtain

$$\hat{B} = -\frac{1}{2}\sqrt{G}R. \quad (2.45)$$

Some insight into the meaning of the connection (2.41) may be gained by examining it in the special gauge where $e^3_i = 0$ and $e^a = (0, 0, 1)$. The remaining components e^a_i may be viewed as a standard frame (zweibein) of a Riemannian 2-manifold, and one finds that ω^a_i is related to conventional Riemannian spin connection by $\omega^a_i = -\frac{1}{2}\epsilon^{abc}\omega^{bc}_i$.

2.7 The Hamiltonian

Our next task is to calculate the expectation value of the Hamiltonian (2.24) in gauge invariant physical states $F[G_{ij}]$. The non-trivial part is the action

of the magnetic field operator (2.20), and we obtain this using the geometric formulae (2.42) together with the functional chain rule

$$\frac{\delta G_{mn}(x)}{\delta E^{bj}(y)} = \epsilon_{mp}\epsilon_{nq} \left(\delta_j^p E^{bq}(x) + \delta_j^q E^{bp}(x) \right) \delta(x-y) \quad (2.46)$$

which follows from (1.2).

We start with the second derivative term in (2.20), which simplifies because $\delta^2 G_{mn}/\delta E^{ai}\delta E^{bj}$ is proportional to δ^{bc} and does not contribute. Hence we get

$$\epsilon^{abc} \frac{\delta^2 F}{\delta E^{bj}\delta E^{ck}} = \epsilon^{abc} \frac{\delta G_{mn}}{\delta E^{bj}} \frac{\delta G_{pq}}{\delta E^{ck}} \frac{\delta^2 F}{\delta G_{mn}\delta G_{pq}}. \quad (2.47)$$

Elementary manipulations using (2.46) and the definition (2.31) of e^a then give

$$-\frac{1}{2}\epsilon^{ij}\epsilon^{abc} \frac{\delta^2 F}{\delta E^{bi}\delta E^{cj}} = -2\sqrt{G} e^a \epsilon_{mp}\epsilon_{nq} \frac{\delta^2 F}{\delta G_{mn}\delta G_{pq}}.$$

Note the determinant-like combination of second derivatives with respect to G_{mn} .

Turning to the gauge covariant derivative term in (2.20), we obtain using (2.31) and (2.46)

$$\epsilon^{ij}\hat{D}_i \frac{\delta F}{\delta E^{aj}} = -2\hat{D}_i \left(e^a{}_j \frac{\delta F}{\delta G_{ij}} \right). \quad (2.48)$$

Using (2.42) we then get

$$\begin{aligned} \epsilon^{ij}\hat{D}_i \frac{\delta F}{\delta E^{aj}} &= -2e^a{}_j \left(\partial_i \frac{\delta F}{\delta G_{ij}} + \Gamma_{ik}^j \frac{\delta F}{\delta G_{ik}} \right) \\ &\equiv -2e^a{}_j \nabla_i \frac{\delta F}{\delta G_{ij}}. \end{aligned} \quad (2.49)$$

It is curious but natural that the use of (2.42), which converts gauge geometry to spatial geometry, automatically brings in the connection terms required for the diffeomorphism covariant divergence of $\frac{\delta F}{\delta G_{ij}}$. This quantity formally transforms as a tensor density of weight one.

Using (2.42) again for \hat{B}^a , we arrive at the geometrical formula for $\bar{B}^a F$:

$$\bar{B}^a F = -2ie^a{}_m \nabla_n \frac{\delta F}{\delta G_{mn}} - \sqrt{G} e^a \left(\frac{1}{2} R F + 2\epsilon_{mp}\epsilon_{nq} \frac{\delta^2 F}{\delta G_{mn}\delta G_{pq}} \right) \quad (2.50)$$

A striking feature of this formula is that its real and imaginary parts are orthogonal in gauge space. A direct consequence is that the energy density

is real, as we will see below. This property was not manifest in the $3 + 1$ dimensional cases treated previously [1], and it should facilitate variational calculations or lattice simulations.

It is now easy to write an explicit expression for expectation values of \overline{H} in (2.24):

$$\langle F | \overline{H} | F \rangle = \int [dG_{ij}] \int d^2x (F[G] | \mathcal{H} | F[G]) \quad (2.51)$$

with functional measure (2.30) and energy density

$$\begin{aligned} (F[G] | \mathcal{H} | F[G]) &= \frac{1}{2} g^2 \delta^{ij} G_{ij} |F|^2 \\ &+ \frac{2}{g^2} G_{ij} \nabla_k \frac{\delta F^*}{\delta G_{ij}} \nabla_\ell \frac{\delta F}{\delta G_{j\ell}} \\ &+ \frac{\det G}{8g^2} \left| R F + 4 \epsilon_{mp} \epsilon_{nq} \frac{\delta^2 F}{\delta G_{mn} \delta G_{pq}} \right|^2. \end{aligned} \quad (2.52)$$

This spatial geometric form of the gauge theory Hamiltonian is our principal result. It is manifestly gauge invariant, real, and local. It is the sum of three positive definite contributions, and the magnetic energy density is singular for configurations where the metric degenerates. The origin of these singularities is the unitary transformation required by the Gauss law constraint, and this is a non-perturbative effect.

3 The singularities of H

The Hamiltonian derived in the previous section is the sum of three real positive terms. The last two terms are the contribution of the magnetic energy density. These terms involve the Christoffel symbol (2.35) and curvature scalar R , which are singular for space-dependent configurations of $G_{ij}(x)$ which are degenerate, i.e. $\det G(x) = 0$. Note that constant degenerate metrics do not make H singular because G^{ij} is always multiplied by $\partial_j G_{k\ell}$ in (2.35). In terms of the electric field $E^{ai}(x)$, a degenerate configuration is entirely regular from a physical standpoint. What happens is that the vectors $E^{a1}(x)$ and $E^{a2}(x)$ become linearly dependent somewhere in space. This is a gauge invariant criterion.

Since the variable $G_{ij}(x)$ is a non-negative 2-tensor, any zero of $\det G(x)$ is generically a local minimum. This fact indicates that the generic case of degeneracy occurs at isolated points of the domain \mathbb{R}^2 . The same conclusion comes from the linear dependence of the $E^{ai}(x)$. Given $E^{a1}(x)$ and $E^{a2}(x)$, the conditions $E^{a1}(x) = c E^{a2}(x)$ constitute three equations to determine

the three quantities x^1 , x^2 , and c . So again one expects that solutions occur at isolated points.

Let us now exemplify the statement in the introduction that a wave functional which is not specially constrained for degenerate fields has infinite energy. Consider the smooth non-covariant functional

$$F[G_{ij}] = \exp \int d^2x \left\{ -\frac{1}{2} G_{11} \nabla^2 G_{11} - \frac{1}{2} G_{22} \nabla^2 G_{22} - \delta^{ij} G_{ij} \right\} \quad (3.1)$$

This is normalizable, since $G_{12}^2 \leq G_{11} G_{22}$, is damped at short wavelengths by the flat Laplacian ∇^2 , and has the unusual feature that no regularization of the second functional derivative term in H is required.

We study the contribution to $\langle F[G] | M | F[G] \rangle$ from diagonal metrics

$$G_{ij}(x) = \begin{pmatrix} \lambda(r) & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.2)$$

a restriction made just to simplify calculations. We assume the C_∞ form $\lambda(r) = r^2 f(r^2)$, with $f(0) \neq 0$ so that $G_{ij}(x)$ is degenerate at $r = 0$. A simple calculation gives the non-vanishing Christoffel symbols and scalar curvature

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2} \frac{x}{r} (\ln \lambda)' & \Gamma_{12}^1 &= \frac{1}{2} \frac{y}{r} (\ln \lambda)' & \Gamma_{11}^2 &= -\frac{1}{2} \frac{y}{r} \lambda' \\ R &= - \left\{ \frac{1}{r} \left(1 - \frac{y^2}{r^2} \right) (\ln \lambda)' - \frac{y^2}{r^2} \left[(\ln \lambda)'' + \frac{1}{2} ((\ln \lambda)')^2 \right] \right\}. \end{aligned} \quad (3.3)$$

The curvature behaves as $1/r^2$ at the origin, so the term

$$\int d^2x (\det G) R^2 F^* F \quad (3.4)$$

gives a logarithmic divergent contribution to the energy which is not canceled elsewhere. (In this case the Christoffel symbols are not singular enough to make the $|\nabla \frac{\partial F}{\partial G}|^2$ terms diverge.)

Of course an integral over all metrics is required to compute the expectation value of the energy $\langle F | H | F \rangle$. It is possible that the infinity found above is irrelevant if the “total functional measure” of degenerate metric configurations vanishes. We will attempt to address this issue in Sec. V, and we now turn to a discussion of the analogous quantum mechanical situation.

4 Quantum Mechanical Central Force Problem for $d = 2$

For reasons that will become clear very quickly, the central force problem in two space dimensions is the relevant quantum mechanical analog for the problem of energy barriers in gauge field theory.

We study

$$\langle \psi | H | \psi \rangle = \int d^2x [\nabla \psi^* \cdot \nabla \psi + V(r) \psi^* \psi] \quad (4.1)$$

for a central potential $V(r)$ which is non-singular at $r = 0$. It is useful to rewrite this in the form

$$\langle \psi | H | \psi \rangle = \int d^2x \left[\frac{1}{2} |(\partial_x + i\partial_y)\psi|^2 + \frac{1}{2} |(\partial_x - i\partial_y)\psi|^2 + V(r) \psi^* \psi \right] \quad (4.2)$$

and introduce the wave function

$$\psi = e^{im\theta} f(r) = \left(\frac{x + iy}{r} \right)^m f(r). \quad (4.3)$$

After calculating derivatives and doing the angular integral, one obtains

$$\langle \psi | H | \psi \rangle = 2\pi \int_0^\infty r dr \left\{ \frac{1}{2} \left| f' - \frac{m}{r} f \right|^2 + \frac{1}{2} \left| f' + \frac{m}{r} f \right|^2 + V(r) |f|^2 \right\}. \quad (4.4)$$

There are two possibly singular barrier terms, and the energy is infinite unless both conditions

$$\lim_{r \rightarrow 0} r \left| f' \mp \frac{m}{r} f \right| = 0 \quad (4.5)$$

hold. This gives only the very weak vanishing condition $f(r) \sim r^\epsilon$.

To obtain a stronger condition we assume that the radial wave function has the product structure $f(r) = f_c(r)R(r)$ where $f_c(r)$ satisfies the equation

$$f'_c - \frac{m}{r} f_c = 0 \quad (4.6)$$

with solution $f_c(r) = r^m$. For $m > 0$ this vanishes at the origin, so the $f' + \frac{m}{r} f$ barrier condition is satisfied in the limit $r \rightarrow 0$. If $m < 0$ the roles of the two conditions are reversed. The net result is the statement that

$$f(r) = f^{|m|} R(r) \quad (4.7)$$

with no constraints on the regular function $R(r)$. Of course we made the extra assumption (4.6) in order to apply the barrier analysis to the first order form of H , but the final result (4.7) agrees with the more rigorous analysis of the second order Schrödinger equation.

One should note that for $d = 3$, the radial measure is $\int r^2 dr$ while the barrier singularity is again $1/r^2$, so it does not seem possible to apply barrier analysis to the first order form of H .

5 The Barrier Functional

In this section we develop the analogy between degenerate configurations of the tensor $G_{ij}(x)$ or the electric field $E^{ai}(x)$, and the singular point $r = 0$ in quantum mechanics. We show that all physical wave functionals have the representation

$$\Psi[E] = e^{i\Omega[E]} F_c[G_{ij}] \mathcal{R}[G_{ij}], \quad (5.1)$$

and define the centrifugal functional $F_c[G_{ij}]$ which “takes care of” the singularities discussed in Sec. 3, either by vanishing for degenerate fields, and/or leaving a less singular functional Schrödinger equation for the residual factor $\mathcal{R}[G_{ij}]$.

The first step is to note that all states which satisfy the Gauss law constraint carry the phase factor $e^{i\Omega[E]}$ which is the analogue of the angular factor $e^{im\theta}$ or $Y_\ell^m(\hat{x})$ for non-zero angular momentum waves in quantum mechanics. These angular functions are singular at $r = 0$, specifically they are not continuous functions of the Cartesian coordinates x, y, z at the origin. The phase $\Omega[E]$ has a similar behavior for degenerate fields which is most easily seen using the following canonical parametrization of the rectangular electric field matrix $E^{ai}(x)$, or, equivalently, its dual e^a_j . If $T^{a1}(x)$ and $T^{a2}(x)$ are two orthogonal 3-vectors, $\mu_1(x)$ and $\mu_2(x)$ non-negative real functions with $\mu_1(x) < \mu_2(x)$, and $\mathcal{R}_{ai} = \cos \theta(x) \delta_{ai} - \sin \theta(x) \epsilon_{ai}$ is a 2×2 orthogonal matrix, then the frame e^a_j , can be expressed as

$$e^a_j(x) = \sum_{\alpha=1}^2 T^{a\alpha}(x) \mu_\alpha(x) \mathcal{R}_{\alpha i}(x), \quad (5.2)$$

a product of “gauge, eigenvalue and spatial rotation” parts. This is essentially the dimensional reduction of the parametrization of the square electric field matrix E^{ai} used in the $3 + 1$ dimensional case in [2]. If we substitute this parametrization into the representation (2.38) for the phase $\Omega[E]$, one

finds

$$\Omega[E] = \int d^2x \epsilon^{ij} \sum_{\alpha=1}^2 e^a (\partial_i T^{a\alpha}) \mu_\alpha \mathcal{R}_{2j} \quad (5.3)$$

with $e^a = \epsilon^{abc} T^{b1} T^{c2}$. At a point of degeneracy, where $\mu_1(x_0) = 0$, the frame behaves as

$$e^a_j(x_0) \longrightarrow T^{a2}(x_0) \mu_2(x_0) \mathcal{R}_{\alpha i}(x_0) \quad (5.4)$$

which is independent of the first row $T^{a1}(x_0)$ of the “gauge matrix,” but the integrand of (5.3) still depends on $T^{a1}(x_0)$. It is this behavior which is qualitatively similar to $e^{i\theta}$ and $Y_\ell^m(\hat{x})$.

The next question we ask is whether $\langle F | H | F \rangle$ in (2.52) is singular enough to permit a “first order barrier analysis.” The clearest way we presently know to address this question is to use a discretization of our Hamiltonian, specifically a rectangular lattice with replacement of spatial derivatives by discrete derivatives. To justify this we recall that one of the principal arguments for a reformulation of non-abelian gauge theory in gauge invariant variables, is that with such variables a cutoff has a gauge invariant meaning. So the crude lattice cutoff we use here should be satisfactory, and we provisionally adopt the attitude that if there is an infinite energy barrier problem in the discretized theory, then it is also a significant issue in the continuum.

It is technically cleaner to study the singularities of H for degenerate metrics, using the parameterization which follows from (5.2), namely

$$G_{ij}(I) = \sum_{\alpha=1,2} \mathcal{R}_{i\alpha}(I) \lambda_\alpha(I) \mathcal{R}_{j\alpha}(I) \quad (5.5)$$

where I refers to the lattice site and $\lambda_\alpha(I) = (\mu_\alpha(I))^2$, $0 \leq \lambda_1(I) \leq \lambda_2(I)$. At each lattice site one has the chain rule and measure

$$\begin{aligned} \frac{\delta}{\delta G_{11}} &= \cos^2 \theta \frac{\delta}{\delta \lambda_1} + \sin^2 \theta \frac{\delta}{\delta \lambda_2} + \frac{\sin \theta \cos \theta}{\lambda_2 - \lambda_1} \frac{\delta}{\delta \theta} \\ \frac{\delta}{\delta G_{22}} &= \sin^2 \theta \frac{\delta}{\delta \lambda_1} + \cos^2 \theta \frac{\delta}{\delta \lambda_2} - \frac{\sin \theta \cos \theta}{\lambda_2 - \lambda_1} \frac{\delta}{\delta \theta} \\ \frac{\delta}{\delta G_{12}} &= \sin \theta \cos \theta \left(\frac{\delta}{\delta \lambda_1} - \frac{\delta}{\delta \lambda_2} \right) + \frac{\cos^2 \theta - \sin^2 \theta}{4(\lambda_2 - \lambda_1)} \frac{\delta}{\delta \theta} \end{aligned} \quad (5.6)$$

$$\prod_{i \leq j} dG_{ij} = (\lambda_2 - \lambda_1) d\lambda_1 d\lambda_2 d\theta.$$

The chain rule can simply be substituted in the lattice Hamiltonian. One can show that the (discretized) Christoffel symbols and scalar curvature behave like $1/\lambda_1(I)$ (to within $\log \lambda_1(I)$ terms) at a point of degeneracy of the configuration $\{G_{ij}(I)\}$ of the discretized metric. Putting things together we find that both the $|\delta F/\delta G|^2$ and $|RF + \delta^2 F/\delta G^2|^2$ terms of H contain the effective barrier singularity $d\lambda_1(I)/\lambda_1(I)$ at each site. This singularity has the same strength as in the $d = 2$ quantum mechanics problem, so the energy is infinite unless wave functions are specially constrained as $\lambda_1(I) \rightarrow 0$ at any site. Specially constrained does not necessarily mean that wave functions vanish, but we prefer to discuss the situation further in the continuum language.

Before doing this we would like to discuss the apparent singularity when $\lambda_1(I) = \lambda_2(I)$ at any lattice site. The net strength of the combined measure and singular terms from the chain rule is $1/|\lambda_2(I) - \lambda_1(I)|$, so there is again a potential infinite energy problem. However we believe that this singularity is an artefact of the choice of variables which is resolved with no physical effect. Specifically the singularity, which originates in the chain rule (5.6), is immediately cancelled if the $\delta/\delta\theta$ derivative acts on functionals $F[G_{ij}]$, where the θ dependence appears only via the metric components in (5.5).

We cannot yet formulate a precise criterion to distinguish between singularities of possible physical significance and those which are just mathematical artifacts. We believe that the physical singularities are those of the phase $\Omega[E]$ which can be expressed as gauge invariant statements about the electric field configuration. An optimal choice of gauge invariant variables is one in which no further singularities appear in the chain rule. This is true for the metric variables $G_{ij}(x)$.

We have reached the conclusion that the singularities of H for degenerate metrics are significant enough to place possibly interesting constraints on wave functionals. Since the situation is similar to $d = 2$ quantum mechanics, we shall try to apply the barrier analysis of Sec. 4 to our Hamiltonian in the “effective first order form” by which it is given in (2.52). We must require that physical wave functionals $F[G]$ satisfy

$$\nabla_j \frac{\partial F}{\partial G_{ij}(x)} = \text{smooth} \quad (5.7)$$

$$\epsilon_{ik}\epsilon_{j\ell} \frac{\partial^2 F}{\partial G_{ij}(x) \partial G_{k\ell}(x)} + \frac{1}{4}R(x)F = \text{smooth} \quad (5.8)$$

where “smooth” means less singular than $\Gamma_{jk}^i(x)$ or $R(x)$ at points where $G_{ij}(x)$ is degenerate.

There is a simple qualitative interpretation of (5.7), since if “= smooth” is replaced by “= 0,” we simply have the condition that $F[G]$ is a diffeomorphism invariant functional of G_{ij} . Of course the full wave functional of the gauge theory cannot be diffeomorphism invariant since H does not have this property. But qualitatively one has the picture of a functional for which the violation of diffeomorphism invariance is “soft” near degenerate metrics. The functional

$$F[G] = \exp - \int d^2x \sqrt{G} [1 + \delta^{ij} G_{ij}] \quad (5.9)$$

is one example. There is also a definite qualitative interpretation of (5.8), which we discuss below, but we note that we have not been able to find any explicit functional which satisfies (5.8). What we have discussed so far is a “conservative” approach to the barrier singularities, and this approach must be called a failure, since it has produced only a weak and vague picture.

Therefore we shall be bolder and postulate the product structure (5.1), with $F_c[G]$ defined as the solution of the equations

$$\nabla_j \frac{\delta F_c}{\delta G_{ij}(x)} = 0 \quad (5.10)$$

$$\epsilon_{ik}\epsilon_{j\ell} \frac{\delta^2 F_c}{\delta G_{ij}(x) \delta G_{k\ell}(x)} + \frac{1}{4} R(x) F_c = 0 \quad (5.11)$$

One question to ask is whether the three conditions are mutually compatible, since it was the incompatibility of the two conditions (4.5) which led to the condition that the radial wave function vanishes at the origin. It turns that (5.8) and (5.9) are compatible, since it is precisely these equations that emerge from a rather different physical context. They are the diffeomorphism and Wheeler-de-Witt constraints of the metric formulation of $2+1$ dimensional general relativity [18] (after continuation of the time coordinate to Euclidean signature). The quantum theory of $2+1$ dimensional gravity has been widely studied, but the usual procedure is to reduce to the finite number of degrees of freedom of a topologically non-trivial compact spatial manifold. Our $F_c[G_{ij}]$ is the unreduced wave functional which is expected to be the unique physical state for the topologically trivial situation of a non-compact spatial 2-surface, and we are not aware of any known explicit solution.

More progress can be made if we consider the equivalent Chern-Simons

[3] (or topological b/F theory [19]) with action

$$\begin{aligned} S &= \frac{1}{2} \int d^3x \epsilon^{\lambda\mu\nu} e_\lambda^a F_{\mu\nu}^a \\ &= \int d^3x \epsilon^{\lambda\mu\nu} e_\lambda^a \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e^{abc} A_\mu^b A_\nu^c \right). \end{aligned} \quad (5.12)$$

One observes that $A_i^a(x)$ and $E^{ai}(x) \equiv \epsilon^{ij} e_j^a(x)$ are canonically conjugate variables which satisfy (2.2), and that the physical state functional $\Psi_c[E]$ in (the unconventional) E -field representation, satisfies the constraint equations

$$D_i E^{ai}(x) \Psi_c[E] = 0 \quad (5.13)$$

$$B^a(x) \Psi_c[E] = \frac{1}{2} \epsilon^{ij} F_{ij}^a(x) \Psi_c[E] = 0. \quad (5.14)$$

The first of these is just the Gauss law constraint (2.3) of the non-abelian gauge theory, while (5.14) is entirely equivalent to (5.10) and (5.11) together. To see this, one need only note from the form of the Hamiltonian that the constraints (5.10) and (5.11) can be interpreted before the unitary transformation (2.11) as the simple statement that the magnetic field $B^a(x)$ annihilate the state $\Psi_c[E] = e^{i\Omega[E]} F_c[G]$. This is just the state which corresponds to the singular object

$$\Psi_c[A] = \prod_{a,x} \delta(B^a(x)) \quad (5.15)$$

in connection representation.

It is a fairly straightforward exercise [20] to show that

$$\begin{aligned} \Psi_c[E] &= \int [dA] \exp \left(-i \int d^2x E^a(x) A_i^a(x) \right) \Psi_c[A] \\ &= \int [dU] \exp \int d^2x \text{Tr} (E^i U^{-1} \partial_i U) \end{aligned} \quad (5.16)$$

where $U(x)$ is a 2×2 $\text{SU}(2)$ matrix and $E^i = \mathcal{T}^a E^{ai}$, with the Pauli matrices \mathcal{T}^a . One can also show by direct functional differentiation that (5.16) satisfies (5.14), essentially because $U^{-1} \partial_i U$ is a “pure gauge.”

The verification of (5.13) is less direct but useful for the further development. We note that the phase $\Omega[E]$ can be written in matrix form as

$$\Omega[E] = -\frac{1}{2} \int d^2x \text{Tr} (E^i \omega_i) \quad (5.17)$$

with $\omega_i = \mathcal{T}^a \omega_i^a$, a form which can be obtained by substitution of (2.37) in (2.38). This means that $\Psi_c[E]$ can be rewritten as

$$\Psi_c[E] = \exp i\Omega[E] \int dU(x) \exp \int d^2x \operatorname{Tr} \left[E^i U^{-1} \left(\partial_i U + \frac{i}{2} U \omega_i \right) \right] \quad (5.18)$$

and the second factor can easily be shown to be invariant under the gauge transformations

$$E^i \longrightarrow V^{-1} E^i V \quad U \longrightarrow UV. \quad (5.19)$$

Thus $\Psi_c[E]$ is the product of the phase factor $\exp i\Omega[E]$ times an explicitly gauge invariant functional of E^i . According to the original argument of Sec. 2, this means that $\Psi_c[E]$ satisfies the Gauss constraint (5.13).

The same argument also tells us that the second factor in (5.18) gives a functional integral representation of the centrifugal functional $F_c[G]$ which satisfies (5.10) and (5.11). Another form of this can be obtained by writing

$$U(x) = \begin{pmatrix} u_1^*(x) & u_2^*(x) \\ -u_2(x) & u_1(x) \end{pmatrix} \quad u_\alpha^* u_\alpha = 1. \quad (5.20)$$

Inserting this in (5.18) we find after some manipulation the representation

$$F_c[G] = \int [du_\alpha(x) du_\beta^*(x) \delta(u_\gamma^*(x) u_\gamma(x) - 1)] \exp \left[-2 \int d^2x u^* E^i D_i u \right] \quad (5.21)$$

where

$$D_i u = \left(\partial_i - \frac{i}{2} \mathcal{T}^a \omega_i^a \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (5.22)$$

This involves a Dirac-like operator and a non-linear measure on the “spinor” fields which reflects the original SU(2) constraints on the matrix $U(x)$. The argument of the exponential in (5.21) is imaginary, since the differential operator is anti-Hermitean,

$$\int d^2x (u^* E^i D_i u)^* = - \int d^2x (u^* E^i D_i u) \quad (5.23)$$

(where (2.26) has been used).

It has not been shown explicitly that (5.21) is a functional of $G_{ij}(x)$, but it follows from the arguments of Sec. 2 that a gauge invariant functional of $E^{ai}(x)$ depends only on $G_{ij}(x)$. Some support for these arguments and the

additional fact that $F_c[G]$ is real comes if we temporarily denote (5.21) by $F_c[E]$. One then has

$$F_c[E]^* = F_c[-E] = F_c[E]. \quad (5.24)$$

The first equality requires only (5.23) and the fact that ω_i is even under $E^i \rightarrow -E^i$, and the second follows from the observation that field configurations $E^i(x)$ and $-E^i(x)$ are related by the gauge transformation which describes a rotation by 180° about the axis $e^a \sim \epsilon^{abc} E^{b1} E^{c2}$. It is also worth observing that any pair of 3-vectors $E^{a1}(x)$ and $E^{a2}(x)$ can be gauge rotated to the 1-2 plane; $E^{31}(x) = E^{32}(x) = 0$ and $e^a(x) = (0, 0, \pm 1)$. The frame $e_i^a = -\epsilon_{ij} E^{aj}$ can then be viewed as a standard zweibein for the metric G_{ij} , and the connection ω_i^a is related to the standard spin connection by $\omega_i^a = -\frac{1}{2} \epsilon^{abc} \omega_i^{bc}$. In this (partially fixed) gauge the differential operator in (5.21) is just the standard Dirac operator for the 2-manifold with zweibein $e_i^a(x)$ and Riemannian spin connection.

We have implicitly assumed that the functional integral representations for $\Psi_c[E]$ and $F_c[G]$ are well defined despite the fact that they involve oscillating integrands, and we hope this is true because $\Psi_c[E]$ is the wave functional of a topological field theory. It is also to be hoped that the knowledge of two-dimensional and topological field theories that has developed during the last decade of work in mathematical physics will lead to some progress toward the evaluation of these path integrals. One approach is to consider a semi-classical approximation constructed using classical solutions of either the metric or topological formulations of 2+1 dimensional gravity. Although the approximation may not satisfy (5.10) or (5.11) exactly, the smoothness conditions (5.7) and (5.8) may be satisfied, and that could be sufficient for the purposes of gauge field theory.

The next step in the exploration of the consequences of the energy barrier is to substitute the product $F[G] = F_c[G] \mathcal{R}[G]$ in the Hamiltonian (2.52) in order to study the effective Hamiltonian governing the residual factor $\mathcal{R}[G]$ in (5.1). Using (5.10) which expresses the diffeomorphic invariance of $F_c[G]$, it is easy to see that

$$\nabla_j \frac{\delta}{\delta G_{j\ell}} (F_c[G] \mathcal{R}[G]) = F_c[G] \nabla_j \frac{\delta \mathcal{R}}{\delta G_{j\ell}}. \quad (5.25)$$

It is also straightforward to apply the second functional derivative to the product, and then use (5.11) to obtain

$$2\epsilon_{ik}\epsilon_{j\ell} \frac{\delta^2 F}{\delta G_{ij} \delta G_{k\ell}} + \frac{1}{2} \mathcal{R} F = 2F_c[G] \epsilon_{ik}\epsilon_{j\ell} \left[\frac{\delta^2 \mathcal{R}}{\delta G_{ij} \delta G_{k\ell}} + 2\rho_{ij} \frac{\delta \mathcal{R}}{\delta G_{k\ell}} \right]$$

$$\rho_{ij}(x) \equiv \frac{\delta}{\delta G_{ij}(x)} \ln F_c[G]. \quad (5.26)$$

We then substitute these results in (2.52) obtain the new form of the Hamiltonian:

$$\begin{aligned} \langle F | H | F \rangle = & \frac{1}{2} \int d^2x \int [dG_{ij}] F_c^2[G] \\ & \left\{ g^2 \delta^{ij} G_{ij} \mathcal{R}^* \mathcal{R} + \frac{4G_{k\ell}}{g^2} \nabla_i \frac{\delta \mathcal{R}^*}{\delta G_{ik}} \nabla_j \frac{\delta \mathcal{R}}{\delta G_{j\ell}} \right. \\ & \left. + \frac{4 \det G}{g^2} \left| \epsilon_{ik} \epsilon_{jl} \left(\frac{\delta^2 \mathcal{R}}{\delta G_{ij} \delta G_{kl}} + 2 \rho_{ij} \frac{\delta \mathcal{R}}{\delta G_{kl}} \right) \right|^2 \right\} \end{aligned} \quad (5.27)$$

Let us discuss this result, noting first that the expression within the brackets $\{ \}$ is less problematic for degenerate fields than the analogous term in (2.52) because the singular quantity RF has been removed. Indeed if the prefactor $F_c[G]^2$ vanishes, then one expects no special constraints on $\mathcal{R}[G]$. However we must also entertain the possibility that $F_c[G]$ does not vanish, for degenerate fields. If its logarithmic derivative $\rho_{ij}(x)$ is also regular, then the only constraint on $\mathcal{R}[G]$ comes from the diffeomorphism term in (5.27). One can avoid an infinite contribution to the energy if, as discussed earlier in this section, $\mathcal{R}[G]$ is a functional with a “soft” violation of diffeomorphism invariance near degenerate metrics. One may also have the situation that $F_c[G]$ is non-vanishing and ρ_{ij} is singular. In this case, it is difficult to be precise, but we expect that the constraint on $\mathcal{R}[G]$ from the $\epsilon_{ij}\epsilon_{j\ell}$ term is less severe than for F itself because there is no longer a singular purely multiplicate term like RF .

Although our investigation has ended in an indefinite way, it is worth summarizing the line of thinking presented in this section. We started by considering the singularities of a formally correct Hamiltonian for a non-abelian gauge theory. Working by analogy with quantum mechanics, we were led to postulate the product structure (5.1) for physical state functionals, and we found that the barrier functionals $\Psi_c[G]$ or $F_c[G]$ have a direct interpretation in a simple topological field theory. It also appears that the factorization of $F_c[G]$ leaves a less singular effective Hamiltonian for $\mathcal{R}[G]$ whether or not $F_c[G]$ vanishes for degenerate fields. The product structure (5.1) is entirely correct, but whether it is useful or not requires further information about the barrier functional $F_c[G]$.

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